Inductive Sequences

Remember that inductively defined sequences are not as scary as they look. Just remember to read $u_{n+1} = 2u_n + 3$ as "The next number in the sequence is twice the previous term plus three". You will also need a starting value like $u_1 = 4$. Therefore this example would yield $4, 11, 25, 53, 109, \ldots$

1. Find the first five terms in the following inductively defined sequences.

   (a) $u_{n+1} = 3u_n - 1$ with $u_1 = 7$. $\boxed{7, 20, 59, 176, 527}$

   (b) $u_{n+1} = 2 - u_n$ with $u_1 = 2$. $\boxed{2, 0, 2, 0, 2}$

   (c) $u_{n+1} = \frac{1}{u_n}$ with $u_1 = 4$. $\boxed{4, \frac{1}{4}, 4, \frac{1}{4}, 4}$

   (d) $u_{n+1} = u_n + 1$ with $u_1 = k$. $\boxed{k, k+1, k+2, k+3, k+4}$

   (e) $u_{n+1} = 2 + \frac{3}{u_n}$ with $u_1 = 1$. $\boxed{1, 5, \frac{13}{5}, \frac{41}{13}, \frac{121}{41}}$

   (f) $u_{n+1} = 2^{u_n} - 1$ with $u_1 = 2$. $\boxed{2, 3, 7, 127, 1.70 \times 10^{38}}$

   (g) $u_{n+1} = 3^{u_n} - 2^{u_n}$ with $u_1 = 2$. $\boxed{2, 5, 211, \ldots}$

   Now with the next term being dependent on the previous two terms...

   (h) $u_{n+1} = u_n + 2u_{n-1}$ with $u_1 = 1$ and $u_2 = 2$. $\boxed{1, 2, 4, 8, 16}$

   (i) $u_{n+1} = \frac{1}{u_n} + \frac{1}{u_{n-1}}$ with $u_1 = -1$ and $u_2 = 3$. $\boxed{-1, 3, -\frac{2}{3}, -\frac{7}{6}, -\frac{33}{14}}$

   (j) $u_{n+1} = \frac{1}{u_n + u_{n-1}}$ with $u_1 = -1$ and $u_2 = 3$. $\boxed{-1, 3, \frac{1}{2}, \frac{2}{7}, \frac{14}{11}}$

   (k) $u_{n+1} = u_n + u_{n-1}$ with $u_1 = a$ and $u_2 = a^2$. $\boxed{a, a^2, a^2 + a, 2a^2 + a, 3a^2 + 2a}$

2. Find the specified term in each sequence (fully simplified, of course).

   (a) $u_{n+1} = \frac{2}{u_n} + 1$ with $u_1 = 3$, find u_4. $\boxed{\frac{21}{11}}$

   (b) $u_{n+1} = \frac{2}{u_n} + 1$ with $u_1 = x$, find u_4. $\boxed{\frac{5x+6}{3x+2}}$

   (c) $u_{n+1} = \frac{1}{u_n + 1}$ with $u_1 = 3$, find u_4. $\boxed{\frac{5}{9}}$

   (d) $u_{n+1} = \frac{2}{u_n + 3}$ with $u_1 = x$, find u_4. $\boxed{\frac{6x+22}{11x+39}}$

   (e) $u_{n+1} = \frac{1+u_n}{1-u_n}$ with $u_1 = 2$, find u_5. $\boxed{2}$

   (f) $u_{n+1} = \frac{1+u_n}{1-u_n}$ with $u_1 = k$, find u_5. $\boxed{k}$

   (g) $u_{n+1} = \frac{2+u_n}{3+2u_n}$ with $u_1 = x$, find u_4. $\boxed{\frac{21x+34}{34x+55}}$

   (h) $u_{n+2} = \frac{1}{u_{n+1} + u_n}$ with $u_1 = x$ and $u_2 = y$, find u_5. $\boxed{\frac{y^3+2xy^2+x^2y+y+x}{2y^2+3xy+x^2+1}}$

3. Find the first five terms of the following sequences. I expect them to be fully cancelled down fractions (or, obviously, whole numbers).

   (a) $u_{n+1} = 2u_n - 1$ with $u_1 = 3$. $\boxed{3, 5, 9, 17, 33}$

   (b) $t_{n+1} = 3t_n + 1$ with $t_1 = \frac{1}{3}$. $\boxed{\frac{1}{3}, 2, 7, 22, 67}$

   (c) $a_{n+1} = 1 - 5a_n$ with $a_1 = 4$. $\boxed{4, -19, 96, -479, 2396}$

   (d) $u_{n+1} = \frac{1}{u_n + 2}$ with $u_1 = 1$. $\boxed{1, \frac{1}{3}, \frac{3}{7}, \frac{7}{17}, \frac{17}{41}}$

   (e) $\theta_{n+1} = \frac{\theta_n + 1}{\theta_n}$ with $\theta_1 = 1$. $\boxed{1, 2, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}}$

   (f) $\psi_{n+1} = 1 + \frac{2}{\psi_n}$ with $\psi_1 = 3$. $\boxed{3, \frac{5}{3}, \frac{11}{5}, \frac{21}{11}, \frac{43}{21}}$

   (g) i. $\alpha_{n+1} = \frac{\alpha_n - 1}{\alpha_n + 1}$ with $\alpha_1 = 5$. $\boxed{5, \frac{2}{3}, -\frac{1}{5}, -\frac{3}{2}, 5}$

   ii. What would the 123^{rd} term of this sequence be? $\boxed{-\frac{1}{5}}$

   (h) $u_{n+1} = 1 + \frac{1}{u_n}$ with $u_1 = k$.